Splitting Functions in the Light Front Hamiltonian Formalism

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In this paper, we have calculated the lowest order splitting functions occurring in the Altarelli-Parisi equation in the light front hamiltonian formalism. Two component perturbative LFQCD in light cone gauge is our guideline and we have used LFTD-approximation for dressed quark and gluon states.

I. INTRODUCTION

It is believed that QCD is the potent theory for the correct description of the strong interaction, but explaining the real situation like the hadrons is still far from being complete. In the low energy non-perturbative sector, it is plagued by the confinement and the vacuum structure problems among others, which must be addressed properly to have the true picture of the QCD bound state. Recently, there has been an increasing amount of interest to explore the non-perturbative domain using the LFQCD in the light cone gauge [1]. Asymptotic freedom and the factorization theorems which separates the soft and hard part of a QCD process, enables one to use the perturbative treatment in the high energy sector. The soft part measures the low energy (nonperturbative) properties of quarks and gluons in the parent hadron and is connected to the parton distribution functions. The hard part is relevant for the scale evolution of the hadronic structure functions and we shall be mainly concerned with that in this paper.

It is a well known fact that the light front current commutators have the bilocal structure in it and the fourier transform of such bilocal matrix element gives the structure function of the deep inelastic scattering [2]. It is just one step further to see that parton pictures emerge directly from it. Scaling of the structure functions is very evident in this calculation. The systematic violation of the scaling comes only when the QCD corrections are taken into account, as can be seen, for example, from the Alteralli-Parisi equation [3].

Parton model is best realised in the infinite momentum frame. Any kind of calculation involving parton model in the equal-time framework assumes some infinite momentum limit (which is sometimes conceptually difficult) and imposes some physical constraint on the gauge field to get the meaningful result. We prefer to attack such problems using light front framework and light cone gauge. Thus, with all the arsenal of two-component perturbative LFQCD in the light cone gauge [4], we study the problem from the first principle without any further assumption. In this paper we calculate the lowest order splitting functions (P_{qq},P_{Gq},P_{qG} and P_{GG}) [3] occurring in the AP-equation following this path. This illustrates the efficiency and both conceptual and calculational simplicity of perturbative LFQCD to produce some physical results.

In this calculation we have used the light front Tamm-Dancoff approximation (LFTD) [5], to describe a dressed particle. This formalism directly gives the probabilistic interpretation of the splitting functions. Similar calculations of the splitting function have been done in the equal time formalism taking infinite momentum limit in the axial gauge [6].

In this paper, we have briefly reviewed the necessary results in the first part and we use those results in our calculation in the latter part of the paper. For detailed discussion one should consult the references mentioned within the text.

II. A BRIEF REVIEW

A. LFQCD hamiltonian

In our calculation we use two component formalism of LFQCD. Here we write the LFQCD hamiltonian as a free term plus the interaction term:

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$$H = \int dx^- d^2x_\perp (\mathcal{H}_0 + \mathcal{H}_{int}) \tag{2.1}$$

where

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2}(\partial^i A^j_a)(\partial^i A^j_a) + \xi^\dagger(\frac{-\partial^2_\perp + m^2}{i\partial^+})\xi \\ \mathcal{H}_{int} &= \mathcal{H}_{qqg} + \mathcal{H}_{ggg} + \mathcal{H}_{qqqq} + \mathcal{H}_{gggg} \end{aligned} \tag{2.2}$$

Here $\mathcal{H}_{int}$ is divided into four parts depending on the nature of the interaction which can be read off from the suffix, for example, $\mathcal{H}_{qqg}$ implies the interaction of two quarks with a gluon and so on. First two terms of the interaction are of the order of g, the coupling constant and the order of the other two is of g^2. Since, in our calculation we shall only be concerned with the lowest order in the coupling constant, we present here the first two terms explicitly.

$$\mathcal{H}_{qqg} = g\xi^\dagger \left\{-2\left(\frac{1}{\partial^+}\right)(\partial_\perp.A_\perp) + (\sigma.A_\perp)\left(\frac{1}{\partial^+}\right)(\sigma.\partial_\perp + m) + \left(\frac{1}{\partial^+}\right)(\sigma.\partial_\perp - m)(\sigma.A_\perp)\right\}\xi \tag{2.3}$$

$$\mathcal{H}_{ggg} = gf^{abc}\left\{(\partial^i A^j_a)A^i_b A^j_c + (\partial^i A^i_a)\left(\frac{1}{\partial^+}\right)(A^j_b \partial^+ A^j_c)\right\} \tag{2.4}$$

Here $A^i_a(x)$'s and $\xi(x)$ stands for the dynamical component of the gauge fields and spinor fields. They are the solutions of the corresponding equation of motion and given by the following expressions,

$$A^i(x) = \sum_\lambda \int \frac{dq^+ d^2q^\perp}{2(2\pi)^3[q+]}\left[\epsilon^i_\lambda a(q,\lambda)e^{-iqx} + h.c\right] \tag{2.5}$$

$$\xi(x) = \sum_\lambda \chi_\lambda \int \frac{dp^+ d^2p^\perp}{2(2\pi)^3}\left[b(p,\lambda)e^{-ipx} + d^\dagger(p,-\lambda)e^{ipx}\right] \tag{2.6}$$

with $q^- = \frac{q^2_\perp}{q^+}$ and $p^- = \frac{p^2_\perp + m^2}{[p^+]}$.

Here λ is defined to be the helicity,

$$\lambda = \begin{cases} 1 \\ -1 \end{cases} \text{for gluons and} \quad \lambda = \begin{cases} \frac{1}{2} \\ -\frac{1}{2} \end{cases} \text{for quarks.} \tag{2.7}$$

The gluon polarization vectors are $\epsilon^i_1 = \frac{1}{\sqrt{2}}(1,i)$ and $\epsilon^i_{-1} = \frac{1}{\sqrt{2}}(1,-i)$ and quark spinors are $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Also the creation and annihilation operators follow the basic commutation relations

$$\begin{aligned} &[a(q,\lambda), a^\dagger(q',\lambda')] = 2(2\pi)^3 q^+ \delta_{\lambda\lambda'}\delta^3(q-q') \\ &\{b(p,\lambda), b^\dagger(p',\lambda')\} = \{d(p,\lambda), d^\dagger(p',\lambda')\} = 2(2\pi)^3\delta_{\lambda,\lambda'}\delta^3(p-p') \end{aligned} \tag{2.8}$$

In our calculations we also require the multiple principal value prescription, given as

$$\begin{aligned} \left(\frac{1}{\partial^+}\right)^n f(x^-) &= \left(\frac{1}{4}\right)^n \int_{-\infty}^{\infty} dx^-_1 dx^-_2 ....dx^-_n \epsilon(x^- - x^-_1)....\epsilon(x^-_{n-1} - x^-_n) f(x^-_n) \\ &\longrightarrow \left[\frac{1}{2}\left(\frac{1}{k^+ + i\epsilon} + \frac{1}{k^+ - i\epsilon}\right)\right]^n f(k^+) \\ &= \frac{1}{[k^+]^n} f(k^+) \end{aligned} \tag{2.9}$$

B. Two component perturbative LFQCD

Here we present the rules that we have used to calculate the necessary matrix elements.
a) Draw all topologically distinct x^+-ordered diagrams.

b) For each vertex, include a factor of $16\pi^3\delta(p_f - p_i)$ and a simple matrix element given below. Each gluon line connected to the vertex contributes a factor $\frac{1}{\sqrt{k^+}}$ from the normalization of the single gluon state.

Necessary matrix elements are as follows.

$$
\begin{aligned}
H_{qqg} &\equiv -gT^a_{\beta\alpha}\chi^\dagger_{\lambda_1}\left\{\frac{2k^i}{[k^+]} - \frac{(\sigma.p_2^\perp - im)}{[p_2^+]}\sigma^i - \sigma^i\frac{(\sigma.p_1^\perp + im)}{[p_1^+]}\right\}\chi_{\lambda_2}\epsilon^{i*}_\lambda \\
&\equiv -gT^a_{\beta\alpha}\chi^\dagger_{\lambda_1}\Gamma^i(p_1, p_2, k)\chi_{\lambda_2}\epsilon^{i*}_\lambda \qquad (2.10)\\
H_{ggg} &\equiv -igf^{abc}\epsilon^i_{\lambda_1}\epsilon^{j*}_{\lambda_2}\epsilon^{l*}_{\lambda_3}\left\{\left[(k_2 - k_3)^i - \frac{k_1^i}{[k_1^+]}(k_2^+ - k_3^+)\right]\delta_{jl}\right. \\
&\quad \left. + \left[(k_3 + k_1)^j - \frac{k_2^j}{[k_2^+]}(k_3^+ + k_1^+)\delta_{il}\right] + \left[-(k^1 + k^2)^l + \frac{k_3^l}{[k_3^+]}(k_1^+ + k_2^+)\right]\delta_{ij}\right\} \\
&\equiv -igf^{abc}\Gamma^{ijl}(k_1, k_2, k_3)\epsilon^i_{\lambda_1}\epsilon^{j*}_{\lambda_2}\epsilon^{l*}_{\lambda_3} \qquad (2.11)
\end{aligned}
$$

Using energy-momentum conserving δ-function and expressing them in terms of relative momentum defined as $x = \frac{k^+}{[p^+]}$, $\kappa_i = k_i - \frac{k^+}{[p^+]}p_i$, we can write

$$
\begin{aligned}
\Gamma^i(p, p-k, k) &= 2\frac{k^i}{[k^+]} - \frac{\sigma^j(p^j - k^j) - im}{[p^+ - k^+]}\sigma^i - \sigma^i\frac{\sigma^j p^j + im}{[p^+]} \\
&= \frac{1}{[p^+][1-x]}\left\{\frac{2}{[x]}\kappa^i - \sigma^i(\sigma.\kappa_\perp) + i\sigma^i mx\right\} \qquad (2.12)\\
\Gamma^{ijl}(p, k, p-k) &= \left[(p - 2k)^i - \frac{p^i}{[p^+]}(p^+ - 2k^+)\right]\delta_{jl} \\
&\quad + \left[(k - 2p)^j - \frac{k^j}{[k^+]}(k^+ - 2p^+)\right]\delta_{il} \\
&\quad + \left[(p + k)^l - \frac{p^l - k^l}{[p^+ - k^+]}(p^+ + k^+)\right]\delta_{ij} \\
&= 2\left\{-\kappa^i\delta_{jl} + \frac{1}{[x]}\kappa^j\delta_{il} + \frac{1}{[1-x]}\kappa^l\delta_{ij}\right\} \qquad (2.13)
\end{aligned}
$$

C. LFTD approximation in (3+1) dimension

Here we briefly review the light front Tamm-Dancoff (LFTD) approximation in (3+1) dimensions which will be necessary for our purpose. LFTD approximation, in its simplest form, is just the Tamm-Dancoff approximation applied to light front field theory. In Tamm-Dancoff approximation we usually truncate the fock space as much as necessary and plausible, and then expand the dressed particle state in the truncated fock space. For example, we can write the state corresponding to a quark as follows:

$$
\begin{aligned}
|quark\rangle &\equiv |\psi(p^+, p^\perp)\rangle \\
&\equiv \frac{c_1(p)}{\sqrt{2(2\pi)^3}}b^\dagger(p,\lambda)|0\rangle \\
&\quad + \sum_{\lambda_1\lambda_2}\int\frac{dq^+d^2q^\perp}{\sqrt{2(2\pi)^3}}\int\frac{dk^+d^2k^\perp}{\sqrt{2(2\pi)^3 k^+}}\delta^3(p - q - k)b^\dagger(q,\lambda_1)a^\dagger(k,\lambda_2)|0\rangle c_2(q,k) \\
&= c_1\psi_1 + c_2\psi_2 \text{ (say)} \qquad (2.14)
\end{aligned}
$$

[Here λ's stand for spin as well as the colour indices.]

Thus we have truncated the fock space to a bare quark plus a quark and a gluon state, so that the maximum occupation number is two. For higher approximation we can take a three particle state also. Here c_1 and c_2 are the probability amplitude of finding a bare quark with all the momentum and of finding a non-interacting pair of quark-gluon with given momentum sharing.

Similarly, we can write a gluon as,

$$\begin{aligned}
|gluon\rangle &\equiv |\psi(p^+, p^\perp)\rangle \\
&\equiv \frac{c_1(p)}{\sqrt{2(2\pi)^3 p^+}} a^\dagger(p,\lambda)|0\rangle \\
&+ \sum_{\lambda_1\lambda_2} \int \frac{dq^+ d^2q^\perp}{\sqrt{2(2\pi)^3}} \int \frac{dk^+ d^2k^\perp}{\sqrt{2(2\pi)^3}} \delta^3(p-q-k) b^\dagger(q,\lambda_1) d^\dagger(k,\lambda_2)|0\rangle c_2(q,k) \\
&+ \frac{1}{2} \sum_{\lambda_1\lambda_2} \int \frac{dq^+ d^2q^\perp}{\sqrt{2(2\pi)^3 q^+}} \int \frac{dk^+ d^2k^\perp}{\sqrt{2(2\pi)^3 k^+}} \delta^3(p-q-k) a^\dagger(q,\lambda_1) a^\dagger(k,\lambda_2)|0\rangle c_2'(q,k)) \\
&\equiv c_1|bare-gluon\rangle + c_2|quark-antiquark\rangle + c_2'|gluon-gluon\rangle. \qquad (2.15)
\end{aligned}$$

We also have to use within this fock space some momentum cutoff and in general we should have a step function such that the invariant mass be less than the cutoff.

One last comment about the normalization of the state. It is to be noted that the state is not yet normalized. Using the normalization condition one can get rid of one of the c's, for example, $c_1(p)$.

III. SPLITTING FUNCTIONS

A. Calculation of P_{qq} and P_{Gq}

First consider the state of a dressed quark according to the Tamm-Dancoff approximation.

$$\begin{aligned}
|\psi(p^+, p^\perp)\rangle &\equiv \frac{c_1(p)}{\sqrt{2(2\pi)^3}} b^\dagger(p,\lambda)|0\rangle + \sum_{\lambda_1\lambda_2} \int \frac{dq^+ d^2q^\perp}{\sqrt{2(2\pi)^3}} \int \frac{dk^+ d^2k^\perp}{\sqrt{2(2\pi)^3 k^+}} \delta^3(p-q-k) \\
&\qquad b^\dagger(q,\lambda_1) a^\dagger(k,\lambda_2)|0\rangle c_2(q,k) \\
&= c_1\psi_1 + c_2\psi_2 \text{ (say)} \qquad (3.1)
\end{aligned}$$

Here λ's stand for spin as well as the colour indices.

Now, we consider the dispersion relation $(P_0^- + P_v^-) = \frac{M^2+P_\perp^2}{P^+}$, and operate this on the quark state.

$$(P_0^- + P_v^-)|\psi\rangle = \frac{M^2+P_\perp^2}{P^+}|\psi\rangle \qquad (3.2)$$

Now take the projection of this equation onto non-interacting one-fermion and one-fermion-one-boson state. This leads to

$$\begin{aligned}
\langle p'|(P_0^- + P_v^-)|\psi\rangle &= \langle p'|\frac{M^2+P_\perp^2}{P^+}|\psi\rangle \\
\langle p'k'|(P_0^- + P_v^-)|\psi\rangle &= \langle p'k'|\frac{M^2+P_\perp^2}{P^+}|\psi\rangle \qquad (3.3)
\end{aligned}$$

where,

$$\begin{aligned}
|p'\rangle &\equiv \frac{1}{\sqrt{2(2\pi)^3}} b^\dagger(p',\lambda_1')|0\rangle \\
|p',k'\rangle &\equiv \frac{1}{\sqrt{2(2\pi)^3}} \frac{1}{\sqrt{2(2\pi)^3 k^{+\prime}}} b^\dagger(p',\lambda_1') a^\dagger(k',\lambda')|0\rangle \qquad (3.4)
\end{aligned}$$

Now, from equation (3.3), we get, up to the lowest order in g,

$$\begin{aligned}
& c_2\langle p'k'|P_0^-|\psi_2\rangle + c_1\langle p'k'|P_v^-|\psi_1\rangle = c_2\langle p'k'|\frac{M^2+P_\perp^2}{P^+}|\psi_2\rangle. \\
\text{or,}\ & c_2\left[\langle p'k'|\frac{M^2+P_\perp^2}{P^+}|\psi_2\rangle - \langle p'k'|P_0^-|\psi_2\rangle\right] = c_1\langle p'k'|P_v^-|\psi_1\rangle. \\
\text{or,}\ & c_2\left[\frac{M^2+(p'_\perp+k'_\perp)^2}{(p'^+ + k'^+)} - \frac{m_F^2+p_\perp'^2}{p'^+} - \frac{m_G^2+k_\perp'^2}{k'^+}\right] = c_1\langle p'k'|P_v^-|\psi_1\rangle. \qquad (3.5)
\end{aligned}$$

Now,$\langle p'k'|P_v^-|\psi_1\rangle$ can easily be calculated using LFQCD perturbation theory. We can use the rule already mentioned and take $M = m_F = m_G = 0$ limit of the above expression. Also, take the total transverse momentum to be zero.

$$\Rightarrow \; p'_\perp + k'_\perp = 0 \text{ or}, \; p'_\perp = -k'_\perp = k_\perp \; (say) \tag{3.6}$$

Thus, we parametrize the momentum as follows:

$$\begin{aligned} \text{total mom. } p &\equiv (p, p, 0_\perp) \\ \text{quark mom. } p' &\equiv \left((1-x)p + \frac{k_\perp^2}{2(1-x)p}, (1-x)p, k_\perp\right) \\ \text{gluon mom. } k' &\equiv \left(xp + \frac{k_\perp^2}{2xp}, xp, -k_\perp\right). \end{aligned} \tag{3.7}$$

With the above prescription, we can write equation (3.5) as,

$$\begin{aligned} c_2\left[-k_\perp^2\left(\frac{1}{x} + \frac{1}{1-x}\right)\right] &= c_1\langle p'k'|P_v^-|\psi_1\rangle \\ \Rightarrow c_2^2\frac{k_\perp^4}{x^2(1-x)^2p^2} &= c_1^2|\langle p'k'|P_v^-|\psi_1\rangle|^2 = \frac{c_1^2}{2(2\pi)^3k'^+}|\langle p'k'|P_v^-|p\rangle|^2. \end{aligned} \tag{3.8}$$

[Note that in the last step, we have written normalization factors and the factor coming out of energy momentum conserving δ-function separately, so that the matrix element is now a number.] Now, to get the correct c_2^2, we sum over all possible intermediate states as we do for the case of an inclusive process and we get,

$$\begin{aligned} c_2^2\frac{k_\perp^4}{x^2(1-x)^2p^2} &= \frac{c_1^2}{2(2\pi)^3k'^+}g^2\sum_{a\beta}\left(T^a_{\beta\alpha}T^a_{\beta\alpha}\right)\chi^\dagger_{\lambda_1}\Gamma^i(p,p',k')\sum_{\lambda_2}\chi_{\lambda_2}\chi^\dagger_{\lambda_2}\Gamma^{\dagger j}\chi_{\lambda_1}\sum_\lambda \epsilon^{i*}_\lambda\epsilon^j_\lambda \\ &= \frac{c_1^2}{2(2\pi)^3k'^+}g^2\Gamma^i(p,p',k')\Gamma^{\dagger i}(p,p',k')\left(\frac{N^2-1}{2N}\right) \end{aligned} \tag{3.9}$$

where,

$$\Gamma^i = \frac{1}{[p^+][1-x]}\left\{\frac{2k^i}{[x]} - \sigma^i(\sigma^j k_\perp^j)\right\} \tag{3.10}$$

$$\Gamma^{\dagger i} = \frac{1}{[p^+][1-x]}\left\{\frac{2k^i}{[x]} - (\sigma^j k_\perp^j)\sigma^i\right\}. \tag{3.11}$$

Here we have used,

$$\sum_{a\beta}\left(T^a_{\beta\alpha}T^a_{\beta\alpha}\right) = \frac{N^2-1}{2N} \tag{3.12}$$

$$\sum_{\lambda_2}\chi_{\lambda_2}\chi^\dagger_{\lambda_2} = 1 \tag{3.13}$$

$$\sum_\lambda \epsilon^{i*}_\lambda\epsilon^j_\lambda = \delta_{ij}. \tag{3.14}$$

Now,

$$\begin{aligned} \Gamma^i\Gamma^{i\dagger} &= \frac{1}{p^2(1-x)^2}\left\{\frac{4k_\perp^2}{x^2} - \frac{4(\sigma^j k_\perp^j)^2}{x} + 2(\sigma^j k_\perp^j)^2\right\} \\ &= \frac{2k_\perp^2}{x^2(1-x)^2p^2}\left[1 + (1-x)^2\right] \end{aligned} \tag{3.15}$$

$$\Rightarrow c_2^2(x, k_\perp) = \frac{c_1^2}{16\pi^3 p}g^2\left(\frac{N^2-1}{2N}\right)\frac{1+(1-x)^2}{x}\frac{2}{k_\perp^2}, \tag{3.16}$$

as $k'^+ = xp$. Integrating over $k^\perp$, we get,

$$c_2^2(x) = \frac{c_1^2}{16\pi^3 p} g^2 \left(\frac{N^2-1}{2N}\right) \frac{1+(1-x)^2}{x} 2\pi \ln\left(\frac{\Lambda}{\mu}\right)^2$$
$$= \frac{c_1^2}{p}\frac{\alpha}{2\pi} P_{Gq} \ln\left(\frac{\Lambda}{\mu}\right)^2 \tag{3.17}$$

where,

$$P_{Gq}(x) = \left(\frac{N^2-1}{2N}\right)\frac{1+(1-x)^2}{x} \tag{3.18}$$

and $\alpha = \frac{g^2}{4\pi}$. Λ and μ are some ultraviolet and infrared cutoff respectively.

P_{qq} can be found using the relation

$$P_{qq}(x) = P_{Gq}(1-x) = \left(\frac{N^2-1}{2N}\right)\frac{1+x^2}{1-x}. \tag{3.19}$$

B. Calculation of P_{qG} and P_{GG}

Now, if we want to calculate P_{qG} or P_{GG}, we have to start with a gluon state which is given by,

$$|gluon\rangle \equiv |\psi(k)\rangle$$
$$\equiv \frac{c_1(k)}{\sqrt{2(2\pi)^3 p^+}} a^\dagger(k,\lambda)|0\rangle$$
$$+ \sum_{\lambda_1\lambda_2} \int \frac{dq^+ d^2q^\perp}{\sqrt{2(2\pi)^3}} \int \frac{dk^+ d^2k^\perp}{\sqrt{2(2\pi)^3}} \delta^3(p-q-k) b^\dagger(q,\lambda_1) d^\dagger(k,\lambda_2)|0\rangle c_2(q,k)$$
$$+\frac{1}{2}\sum_{\lambda_1\lambda_2} \int \frac{dk_1^+ d^2k_1^\perp}{\sqrt{2(2\pi)^3 k_1^+}} \int \frac{dk_2^+ d^2k_2^\perp}{\sqrt{2(2\pi)^3 k_2^+}} \delta^3(k-k_1-k_2) a^\dagger(k_1,\lambda_1) a^\dagger(k_2,\lambda_2)|0\rangle c_2'(k_1,k_2) \tag{3.20}$$

Then we proceed along the same way as before. Consider the eigenvalue equation for the gluon state and take the projection onto a quark-antiquark state and two-gluon state separately.

$$|quark - antiquark\rangle \equiv |p'q'\rangle \equiv \frac{1}{\sqrt{2(2\pi)^3}} b^\dagger(p',\lambda_1') d^\dagger(q',\lambda_2')|0\rangle \tag{3.21}$$

$$|two - gluon\rangle \equiv |k_1', k_2'\rangle \equiv \frac{1}{\sqrt{2(2\pi)^3 k_1^{+\prime}}} \frac{1}{\sqrt{2(2\pi)^3 k_2^{+\prime}}} a^\dagger \dagger(k_1',\lambda_1') a^\dagger(k_2',\lambda_2')|0\rangle \tag{3.22}$$

In the first case, 1st and 2nd term from the gluon state will contribute and we get as in equation (3.8),

$$c_2^2 \frac{k_\perp^4}{x^2(1-x)^2 k^2} = c_1^2 |\langle p'q'|P_v^-|\psi_1\rangle|^2 = \frac{c_1^2}{2(2\pi)^3 k} |\langle p'q'|P_v^-|k\rangle|^2. \tag{3.23}$$

Here, we have used the following momentum parametrization.

$$\text{total gluon mom. } k \equiv (k, k, 0_\perp)$$
$$\text{antiquark mom. } q' \equiv \left((1-x)k + \frac{k_\perp^2}{2(1-x)k}, (1-x)k, -k_\perp\right)$$
$$\text{gluon mom. } p' \equiv \left(xk + \frac{k_\perp^2}{2xk}, xk, k_\perp\right). \tag{3.24}$$

Now, with this parametrization $\langle p'q'|P_v^-|k\rangle$ is exactly identical to $\langle p'q'|P_v^-|p\rangle$ we already have calculated, with the difference that Γ^i now becomes

$$\Gamma^i = \frac{1}{[k^+][1-x]}\left\{2k^i - \frac{\sigma^i(\sigma^j k_\perp^j)}{[x]}\right\} \tag{3.25}$$

$$\Gamma^{i\dagger} = \frac{1}{[k^+][1-x]}\left\{2k^i - \frac{(\sigma^j k_\perp^j)}{[x]}\sigma^i\right\} \tag{3.26}$$

Now, summing over all possible intermediate states, we get,

$$\begin{aligned}\sum |\langle p'q'|P_v^-|k\rangle|^2 &= g^2 \sum_{\beta\alpha}\left(T^a_{\beta\alpha}T^a_{\beta\alpha}\right)\sum_{\lambda_1\lambda_2}\left(\chi^\dagger_{\lambda_1}\Gamma^i\chi_{\lambda_2}\chi^\dagger_{\lambda_2}\Gamma^{j\dagger}\chi_{\lambda_1}\right)\epsilon^{i*}_\lambda\epsilon^j_\lambda \\ &= g^2\left(\frac{1}{2}\right)2\Gamma^i\Gamma^{j\dagger}\epsilon^{i*}_\lambda\epsilon^j_\lambda \end{aligned} \tag{3.27}$$

[The factor 2 comes from the summation over λ_1.]

At this point we assume that the result is independent of initial gluon polarizations and hence we add the result for different polarization and divide by two to get the final result.

$$\rightarrow \sum |\langle p'q'|P_v^-|k\rangle|^2 = \frac{1}{2}g^2\Gamma^i\Gamma^{i\dagger} \tag{3.28}$$

[As $\sum_\lambda \epsilon^{i*}_\lambda\epsilon^j_\lambda = \delta_{ij}$.]
Now,

$$\Gamma^i\Gamma^{i\dagger} = \frac{1}{k^2(1-x)^2}\left\{4k_\perp^2 - \frac{4(\sigma.k_\perp)^2}{x} + \frac{2(\sigma.k_\perp)^2}{x^2}\right\} = \frac{2k_\perp^2}{k^2x^2(1-x)^2}[x^2+(1-x)^2]. \tag{3.29}$$

$$\Rightarrow\ c_2^2(x,k_\perp) = \frac{c_1^2}{16\pi^3 k}\frac{1}{2}g^2[x^2+(1-x)^2]\frac{2}{k_\perp^2} \tag{3.30}$$

Integration over $k_\perp$ gives,

$$c_2^2 = \frac{c_1^2}{k}P_{qG}\frac{\alpha}{2\pi}\ln\left(\frac{\Lambda}{\mu}\right)^2 \tag{3.31}$$

where,

$$P_{qG} = \frac{1}{2}[x^2+(1-x)^2]. \tag{3.32}$$

In the second case, when we take the projection onto two-gluon state, 1st and 3rd term in equation (3.20) will contribute, and we get as in equation (3.8),

$$\begin{aligned} c_2^{2\prime}\frac{k_\perp^4}{x^2(1-x)^2k^2} &= c_1^2|\langle k_1'k_2'|P_v^-|\psi_1\rangle|^2 \\ &= \frac{c_1^2}{2(2\pi)^3}\frac{1}{k^3(1-x)x}|\langle k_1'k_2'|P_v^-|k\rangle|^2. \end{aligned} \tag{3.33}$$

[Again, we have extracted the factors arising from the normalization of states and the energy-momentum conserving δ-function, and write them separately so that the remaining factor is just a number.]

Now, summing over all the intermediate gluon states and averaging over the initial gluon states, we get the following result.

$$\begin{aligned}\overline{\sum}|\langle q,k|P_v^-|p\rangle|^2 = g^2\left(\sum_{bc}f^{abc}f_{abc}\right)\sum_{\substack{i\ \ j\ \ l\\ i'\ j'\ l'}}\Gamma^{ijl}\Gamma^{\dagger i'j'l'} \\ \frac{1}{2}\sum_{\lambda_1}\epsilon^{i*}_{\lambda_1}\epsilon^{i'}_{\lambda_1}\sum_{\lambda_2}\epsilon^{j*}_{\lambda_2}\epsilon^{j'}_{\lambda_2}\sum_{\lambda_3}\epsilon^{l*}_{\lambda_3}\epsilon^{l'}_{\lambda_3}\end{aligned} \tag{3.34}$$

Now using the relations

$$\sum_{bc} f^{abc} f_{abc} = N \tag{3.35}$$

$$\sum_{\lambda_2} \epsilon^{i*}_{\lambda_2} \epsilon^{i\prime}_{\lambda_2} = \delta_{ii'}. \tag{3.36}$$

$$\Rightarrow \overline{\sum} |\langle q,k|P_v^-|p\rangle|^2 = g^2 N \frac{1}{2} \sum_{ijl} \Gamma^{ijl}\Gamma^{ijl\dagger} \tag{3.37}$$

Now,

$$\Gamma^{ijl}\Gamma^{ijl\dagger} = 8k_\perp^2 \left[1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right]. \tag{3.38}$$

$$\rightarrow \overline{\sum} |\langle q,k|P_v^-|p\rangle|^2 = g^2 N \left[1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right] 4k_\perp^2 \tag{3.39}$$

Thus, using this result, we get,

$$c_2^{2\prime}(x,k_\perp) = \frac{c_1^2}{k}\left[x(1-x) + \frac{x}{1-x} + \frac{1-x}{x}\right](2N)\left(\frac{g^2}{16\pi^3}\right)\left(\frac{2}{k_\perp^2}\right). \tag{3.40}$$

Integration over $k_\perp$ gives,

$$c_2^{2\prime} = \frac{c_1^2}{k} P_{GG} \frac{\alpha}{2\pi} \ln\left(\frac{\Lambda}{\mu}\right)^2 \tag{3.41}$$

where,

$$P_{GG} = 2N\left[x(1-x) + \frac{x}{(1-x)} + \frac{1-x}{x}\right]. \tag{3.42}$$

IV. CONCLUSION

Using the normalization condition we can write $c_1^2 = 1$, in the equations (3.17), (3.31), and (3.41), since we are working only with the lowest order contributions in the coupling constant. Thus, we can see directly that the splitting functions are proportional to the corresponding probabilities as is the case in AP-equation. Equation (3.18), (3.19), (3.32) and (3.42) show the expressions for the splitting functions which are exactly identical to that derived in AP-paper.

Of course, we have not considered the contributions coming from the infrared singularity ($x \rightarrow 1$ limit) in the expression of splitting function. One usually puts them on the basis of some physical arguments or by direct loop calculations.

In principle one should be able starting from the first principle, to derive the AP-equation in the hamiltonian framework. The very success of our calculation in obtaining the splitting functions suggests that it may be plausible to try and extrapolate it to obtain AP-equation, which is, of course, a non-trivial job.

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